CONJUGATE NETS R AND THEIR TRANSFORMATIONS.*

By Luther Pfahler Eisenhart.

1. A rectilinear congruence for which the asymptotic lines on the two focal surfaces correspond is called a W-congruence. When the tangents to the curves of each family of a conjugate system of curves on a surface form W congruences, the system is called a net R.† It is the purpose of this paper to establish two types of transformations of an R net into R nets, called *transformations* W and *transformations* T.

If N is an R net, each pair of solutions of two completely integrable partial differential equations of the second order determine a W transform $\bar{N}$ which is an R net; the nets N and $\bar{N}$ are on the focal surfaces of a W-congruence, and either net is a *derived net* of the other, in the sense of Guichard. These transformations W admit a theorem of permutability, that is if $\bar{N}_1$ and $\bar{N}_2$ are W transforms of N, there exists an R net $\bar{N}_{12}$ which is a W transform of $\bar{N}_1$ and $\bar{N}_2$.

In a previous paper‡ the author established a theory of *transformations* T of any net whatsoever such that if N_1 is a T transform of a net N, the developables of the congruence of lines joining corresponding points of N and N_1 meet the two surfaces on which N and N_1 lie in these nets. In § 8 it is shown that an R net admits a group of transformations T into R nets. Moreover, these transformations admit theorems of permutability similar to W transformations.

In § 10 it is shown that if $\bar{N}$ and N_1 are W and T transforms respectively of an R net N, there exists a net $\bar{N}_1$ which is a T transform of $\bar{N}$ and a W transform of N_1.

In another paper§ we apply these results to the surfaces applicable to a quadric and show that the transformations B_k of these surfaces established by Bianchi‖ are of the type W, and that the transformations of Guichard¶ are of the type T.

2. **Differential equations of a net.** If x, y, z are the cartesian coördinates

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† Tzitzeica, Comptes Rendus, vol. 152 (1911), p. 1077; also Demoulin, Comptes Rendus, vol. 153 (1912), p. 590.

‡ Transactions of the American Mathematical Society, vol. 18 (1917), pp. 99–124. This paper will be referred to as M.

§ Proceedings of the Strasbourg Congress.

‖ Lezioni, vol. 3.

¶ Mémoires á L'Academie des Sciences, vol. 34 (1909).

of a surface S upon which the parametric curves form a conjugate system or net, the coördinates are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial \theta}{\partial v}, \tag{1}$$

which we call the point equation of the net N. This follows from one of the three *equations of Gauss* for S.* From the other two of these equations it follows that x, y and z are solutions also of an equation of the form

$$\frac{\partial^2\theta}{\partial v^2} = r\frac{\partial^2\theta}{\partial u^2} + a'\frac{\partial \theta}{\partial u} + b'\frac{\partial \theta}{\partial v}, \tag{2}$$

where

$$r = \frac{D''}{D}, \tag{3}$$

the functions D and D'' being second fundamental coefficients of S.

In order that two equations of the form (1) and (2) admit three independent solutions, it is necessary and sufficient that the functions a, b, a', b' and r satisfy three differential equations of condition. Instead of calculating these conditions, we determine the conditions that the more general system

$$\begin{aligned} \frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial \theta}{\partial v} + c\theta, \\ \frac{\partial^2\theta}{\partial v^2} &= r\frac{\partial^2\theta}{\partial u^2} + a'\frac{\partial \theta}{\partial u} + b'\frac{\partial \theta}{\partial v} + c'\theta \end{aligned} \tag{4}$$

admit four independent solutions.

Equating the expressions for $\partial^3\theta/\partial u\partial v^2$ obtained from these equations by differentiation, we have

$$\frac{\partial^3\theta}{\partial u^3} = A_1\frac{\partial^2\theta}{\partial u^2} + B_1\frac{\partial \theta}{\partial u} + C_1\frac{\partial \theta}{\partial v} + D_1\theta, \tag{5}$$

where

$$\begin{aligned} A_1 &= \frac{\partial}{\partial u}\log\frac{b}{r} - \frac{a'}{r}, \\ B_1 &= \frac{1}{r}\left(\frac{1}{a}\frac{\partial^2 a}{\partial v^2} + a'\frac{\partial}{\partial u}\log\frac{b}{a'} - b'\frac{\partial \log a}{\partial v} - c'\right), \\ C_1 &= \frac{1}{r}\left(\frac{\partial^2 \log b}{\partial u\partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c - \frac{\partial b'}{\partial u}\right), \\ D_1 &= \frac{1}{r}\left(c\frac{\partial \log a}{\partial v} + c'\frac{\partial \log b}{\partial u} + \frac{\partial c}{\partial v} - cb' - \frac{\partial c'}{\partial u}\right). \end{aligned} \tag{6}$$

* Cf. Eisenhart, Differential Geometry, p. 154. Hereafter a reference to this book will be written, E., p. 154.

Differentiating the first of (4) with respect to u, we obtain

$$\frac{\partial^3\theta}{\partial u^2\partial v} = A_2\frac{\partial^2\theta}{\partial u^2} + B_2\frac{\partial\theta}{\partial u} + C_2\frac{\partial\theta}{\partial v} + D_2\theta, \tag{7}$$

where

$$\begin{aligned} A_2 &= \frac{\partial \log a}{\partial v}, & B_2 &= \frac{\partial^2 \log a}{\partial u\partial v} + \frac{\partial \log a}{\partial v}\frac{\partial \log b}{\partial u} + c, \\ C_2 &= \frac{1}{b}\frac{\partial^2 b}{\partial u^2}, & D_2 &= c\frac{\partial}{\partial u}\log bc. \end{aligned} \tag{8}$$

When from (5) and (7) we express the condition

$$\frac{\partial}{\partial v}\left(\frac{\partial^3\theta}{\partial u^3}\right) = \frac{\partial}{\partial u}\left(\frac{\partial^3\theta}{\partial u^2\partial v}\right),$$

we get an equation of the form

$$P\frac{\partial^2\theta}{\partial u^2} + Q\frac{\partial\theta}{\partial u} + R\frac{\partial\theta}{\partial v} + S\theta = 0, \tag{9}$$

where P, Q, R and S are determinate functions. If we do not have

$$P = Q = R = S = 0, \tag{10}$$

equations (4) and (9) admit at most three independent solutions. Hence (10) must hold. When their expressions are calculated we find that (10) is equivalent to

$$\begin{aligned} &\frac{\partial A_1}{\partial v} + C_1 r = \frac{\partial A_2}{\partial u} + B_2, \\ &\frac{\partial B_1}{\partial v} + A_1B_2 + a'C_1 = \frac{\partial B_2}{\partial u} + C_2\frac{\partial \log a}{\partial v} + D_2, \\ &\frac{\partial C_1}{\partial v} + A_1C_2 + B_1\frac{\partial \log b}{\partial u} + b'C_1 + D_1 = \frac{\partial C_2}{\partial u} + A_2C_1 + C_2\frac{\partial \log b}{\partial u}, \\ &\frac{\partial D_1}{\partial v} + A_1D_2 + B_1c + C_1c' = \frac{\partial D_2}{\partial u} + A_2D_1 + C_2c. \end{aligned} \tag{11}$$

It is readily seen that when these conditions are satisfied, equations (4), (5) and (7) are consistent, and consequently equations (4) admit four independent solutions.

When we take equations (1) and (2) in place of (4), we have $c = c' = D_1 = D_2 = 0$, and the last of (11) is satisfied identically.

3. **Equations of a net R.** When the tangents are drawn to the curves $v = \text{const.}$, the second focal net of the congruence of tangents is given by

$$x_{-1} = x - \frac{1}{\frac{\partial \log b}{\partial u}} \frac{\partial x}{\partial u}.\text{*} \tag{12}$$

By differentiation we have

$$\begin{aligned} \frac{\partial x_{-1}}{\partial u} &= \frac{C_2}{\left(\frac{\partial \log b}{\partial u}\right)^2} \frac{\partial x}{\partial u} - \frac{1}{\frac{\partial \log b}{\partial u}} \frac{\partial^2 x}{\partial u^2}, \\ \frac{\partial x_{-1}}{\partial v} &= -K \frac{1}{\left(\frac{\partial \log b}{\partial u}\right)^2} \frac{\partial x}{\partial u}, \end{aligned} \tag{13}$$

where

$$K = -\frac{\partial^2 \log b}{\partial u \partial v} + \frac{\partial \log a}{\partial v} \frac{\partial \log b}{\partial u}. \tag{14}$$

With the aid of (5) we find

$$\begin{aligned} \frac{\partial^2 x_{-1}}{\partial u^2} &= \left(A_1 - \frac{C_2}{\frac{\partial \log b}{\partial u}} - \frac{\frac{\partial^2 \log b}{\partial u^2}}{\frac{\partial \log b}{\partial u}} \right) \frac{\partial x_{-1}}{\partial u} \\ &\quad + \frac{1}{\frac{\partial \log b}{\partial u}} \frac{\partial x}{\partial u} \left[\frac{\partial}{\partial u} \left(\frac{C_2}{\frac{\partial \log b}{\partial u}} \right) - B_1 + \left(\frac{C_2}{\frac{\partial \log b}{\partial u}} - A_1 \right) \frac{C_2}{\frac{\partial \log b}{\partial u}} \right] \\ &\quad - \frac{C_1}{\frac{\partial \log b}{\partial u}} \frac{\partial x}{\partial v}, \\ \frac{\partial^2 x_{-1}}{\partial v^2} &= \frac{\partial}{\partial v} \log \frac{K}{\left(\frac{\partial \log b}{\partial u}\right)^2} \frac{\partial x_{-1}}{\partial v} - \frac{K}{\left(\frac{\partial \log b}{\partial u}\right)^2} \\ &\qquad \times \left(\frac{\partial \log a}{\partial v} \frac{\partial x}{\partial u} + \frac{\partial \log b}{\partial u} \frac{\partial x}{\partial v} \right). \end{aligned} \tag{15}$$

From (3) it follows that if the asymptotic lines are to correspond on the two focal surfaces, we must have an equation of the form

$$\frac{\partial^2 x_{-1}}{\partial v^2} = r \frac{\partial^2 x_{-1}}{\partial u^2} + a_{-1}' \frac{\partial x_{-1}}{\partial u} + b_{-1}' \frac{\partial x_{-1}}{\partial v}. \tag{16}$$

* E., p. 405.

From (13) and (16) it follows that a necessary and sufficient condition is that $C_1 r = K$, or in consequence of (6),

$$\frac{\partial b'}{\partial u} = 2\frac{\partial^2 \log b}{\partial u \partial v}. \tag{17}$$

In like manner the condition that the tangents to the curves $u =$ const. of N from a W congruence is

$$\frac{\partial a'}{\partial v} = 2\frac{\partial^2 \log a}{\partial u \partial v}. \tag{18}$$

Hence equations (17) and (18) constitute the condition that N be an R net. When these conditions are satisfied we have from the first of (11) that $\partial^2 \log r/\partial u \partial v = 0$. Consequently we have the theorem of Tzitzeica:

An R net is isothermal conjugate.

Accordingly the parameters u and v can be chosen so that $r = -1$. Since a and b in (1) are determined only to within factors which are functions of u and v respectively, we have the theorem:

The two differential equations satisfied by the cartesian coördinates of an R net are reducible to the forms

$$\begin{aligned} \frac{\partial^2 \theta}{\partial u \partial v} &= \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial \theta}{\partial v}, \\ \frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} &= 2\frac{\partial \log a}{\partial u}\frac{\partial \theta}{\partial u} + 2\frac{\partial \log b}{\partial v}\frac{\partial \theta}{\partial v}. \end{aligned} \tag{19}$$

The complete determination of R nets requires the solution of the two equations to be satisfied by a and b which follow from the second and third of (11).

4. **Derived nets.** If N is a net whose point equation is (1), the equations

$$\begin{gathered} \frac{\partial x'}{\partial u} = h\frac{\partial x}{\partial u}, \quad \frac{\partial x'}{\partial v} = l\frac{\partial x}{\partial v}; \quad \frac{\partial y'}{\partial u} = h\frac{\partial y}{\partial u}, \quad \frac{\partial y'}{\partial v} = l\frac{\partial y}{\partial v}, \\ \frac{\partial z'}{\partial u} = h\frac{\partial z}{\partial u}, \quad \frac{\partial z'}{\partial v} = l\frac{\partial z}{\partial v} \end{gathered} \tag{20}$$

are consistent provided h and l satisfy the equations

$$\frac{\partial h}{\partial v} = (l - h)\frac{\partial \log a}{\partial v}, \qquad \frac{\partial l}{\partial u} = (h - l)\frac{\partial \log b}{\partial u}. \tag{21}$$

It is readily found that x', y', z' satisfy the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial}{\partial v}\log ah\frac{\partial \theta}{\partial u} + \frac{\partial}{\partial u}\log bl\frac{\partial \theta}{\partial v}, \tag{22}$$

and consequently are the cartesian coördinates of a net N' *parallel* to N, as follows from the form of (20). Moreover, whenever the cartesian coördinates of two surfaces satisfy equations of the form (20), the tangents to the parametric curves at corresponding points are parallel, and these curves form nets on the two surfaces.

When the lines of a congruence H lie in the tangent planes of a net N, the developables of H correspond to the curves of N and the focal points of H lie on the tangents to the curves of N, the congruence H is said to be *harmonic* to N. Each solution θ of (1) determines such a harmonic congruence. The coördinates of the foci are of the forms

$$x - \frac{\theta}{\frac{\partial \theta}{\partial u}} \frac{\partial x}{\partial u}, \qquad x - \frac{\theta}{\frac{\partial \theta}{\partial v}} \frac{\partial x}{\partial v}.* \tag{23}$$

If H_1 and H_2 are two congruences harmonic to N determined by solutions θ_1 and θ_2 of (1), corresponding lines of H_1 and H_2 meet in a point $\bar{M}$ whose coördinates are of the form

$$\bar{x} = x + p\frac{\partial x}{\partial u} + q\frac{\partial x}{\partial v}, \tag{24}$$

where

$$\begin{gathered} p = \frac{1}{\Delta}\left(\theta_1 \frac{\partial \theta_2}{\partial v} - \theta_2 \frac{\partial \theta_1}{\partial v}\right), \qquad q = \frac{1}{\Delta}\left(\theta_2 \frac{\partial \theta_1}{\partial u} - \theta_1 \frac{\partial \theta_2}{\partial u}\right), \\ \Delta = \frac{\partial \theta_2}{\partial u}\frac{\partial \theta_1}{\partial v} - \frac{\partial \theta_2}{\partial v}\frac{\partial \theta_1}{\partial u}. \end{gathered} \tag{25}$$

By differentiation we have

$$\begin{aligned} \frac{\partial \bar{x}}{\partial u} &= p\left[\frac{\partial^2 x}{\partial u^2} + \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial u^2}\frac{\partial \theta_2}{\partial v} - \frac{\partial^2 \theta_2}{\partial u^2}\frac{\partial \theta_1}{\partial v}\right)\frac{\partial x}{\partial u} \right. \\ &\qquad \left. - \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial u^2}\frac{\partial \theta_2}{\partial u} - \frac{\partial^2 \theta_2}{\partial u^2}\frac{\partial \theta_1}{\partial u}\right)\frac{\partial x}{\partial v}\right], \\ \frac{\partial \bar{x}}{\partial v} &= q\left[\frac{\partial^2 x}{\partial v^2} + \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial v^2}\frac{\partial \theta_2}{\partial v} - \frac{\partial^2 \theta_2}{\partial v^2}\frac{\partial \theta_1}{\partial v}\right)\frac{\partial x}{\partial u} \right. \\ &\qquad \left. - \frac{1}{\Delta}\left(\frac{\partial^2 \theta_1}{\partial v^2}\frac{\partial \theta_2}{\partial u} - \frac{\partial^2 \theta_2}{\partial v^2}\frac{\partial \theta_1}{\partial u}\right)\frac{\partial x}{\partial v}\right]. \end{aligned} \tag{26}$$

It is readily seen that the functions θ_1' and θ_2' defined by

$$\frac{\partial \theta_1'}{\partial u} = h\frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = l\frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta_2'}{\partial u} = h\frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2'}{\partial v} = l\frac{\partial \theta_2}{\partial v} \tag{27}$$

are solutions of (22), the point equation of the net N' parallel to N as

* Guichard, Annales de L'Ecole Normale, Ser. 3, vol. 14 (1897).

given by (20). We call θ_1 and θ_1' *corresponding solutions* of (1) and (22); likewise θ_2 and θ_2'.

By means of θ_1' and θ_2' we obtain congruences H_1' and H_2' harmonic to N', and corresponding lines meet in $\bar{M}'$ whose coördinates are

$$\bar{x}' = x' + p'\frac{\partial x'}{\partial u} + q'\frac{\partial x'}{\partial v}, \tag{28}$$

where p' and q' are analogous to (25) in θ_1' and θ_2'. The derivatives of $\bar{x}'$ are expressible by means of the preceding formulas in the forms

$$\frac{\partial \bar{x}'}{\partial u} = \frac{\theta_2'\dfrac{\partial \theta_1}{\partial v} - \theta_1'\dfrac{\partial \theta_2}{\partial v}}{\theta_2\dfrac{\partial \theta_1}{\partial v} - \theta_1\dfrac{\partial \theta_2}{\partial v}}\frac{\partial \bar{x}}{\partial u}, \qquad \frac{\partial \bar{x}'}{\partial v} = \frac{\theta_2'\dfrac{\partial \theta_1}{\partial u} - \theta_1'\dfrac{\partial \theta_2}{\partial u}}{\theta_2\dfrac{\partial \theta_1}{\partial u} - \theta_1\dfrac{\partial \theta_2}{\partial u}}\frac{\partial \bar{x}}{\partial v}. \tag{29}$$

Since these equations are of the form (20), the points $\bar{M}$ and $\bar{M}'$ describe nets, $\bar{N}$ and $\bar{N}'$, which are parallel to one another. Guichard* calls $\bar{N}$ a derived net of N.

Since θ_1 and θ_2 are solutions of (1), we have

$$\begin{aligned}
\frac{\partial p}{\partial u} &= -1 - q\frac{\partial \log a}{\partial v} + \frac{p}{\Delta}\left(\frac{\partial^2\theta_1}{\partial u^2}\frac{\partial \theta_2}{\partial v} - \frac{\partial^2\theta_2}{\partial u^2}\frac{\partial \theta_1}{\partial v}\right),\\
\frac{\partial p}{\partial v} &= -p\frac{\partial \log a}{\partial v} + \frac{1}{\Delta}\left(\theta_1\frac{\partial^2\theta_2}{\partial v^2} - \theta_2\frac{\partial^2\theta_1}{\partial v^2}\right) - \frac{p}{\Delta}\left(\frac{\partial^2\theta_1}{\partial v^2}\frac{\partial \theta_2}{\partial u} - \frac{\partial^2\theta_2}{\partial v^2}\frac{\partial \theta_1}{\partial u}\right),\\
\frac{\partial q}{\partial u} &= -q\frac{\partial \log b}{\partial u} + \frac{1}{\Delta}\left(\theta_2\frac{\partial^2\theta_1}{\partial u^2} - \theta_1\frac{\partial^2\theta_2}{\partial u^2}\right) - \frac{q}{\Delta}\left(\frac{\partial^2\theta_2}{\partial u^2}\frac{\partial \theta_1}{\partial v} - \frac{\partial^2\theta_1}{\partial u^2}\frac{\partial \theta_2}{\partial v}\right),\\
\frac{\partial q}{\partial v} &= -1 - p\frac{\partial \log b}{\partial u} + \frac{q}{\Delta}\left(\frac{\partial^2\theta_2}{\partial v^2}\frac{\partial \theta_1}{\partial u} - \frac{\partial^2\theta_1}{\partial v^2}\frac{\partial \theta_2}{\partial u}\right).
\end{aligned} \tag{30}$$

With the aid of these expressions we show that the point equation of $\bar{N}$ is

$$\frac{\partial^2\bar{\theta}}{\partial u\partial v} = \frac{\partial}{\partial v}\log ap\frac{\partial \bar{\theta}}{\partial u} + \frac{\partial}{\partial u}\log bq\frac{\partial \bar{\theta}}{\partial v}. \tag{31}$$

5. **Reciprocally derived nets.** Ordinarily N is not a derived net of $\bar{N}$. When it is, we say that N is a *reciprocally derived* net. Tzitzeica† was the first to consider nets of this kind. If N is a derived net of $\bar{N}$, the tangent planes of $\bar{N}$ pass through corresponding points of N just as the tangent planes of N pass through the corresponding points of $\bar{N}$. Hence the surfaces S and $\bar{S}$ on which N and $\bar{N}$ lie are the focal surfaces of the congruence $\bar{G}$ of lines joining corresponding points. Since the lines of $\bar{G}$

* L. c., p. 489.

† Comptes Rendus, vol. 156 (1913), p. 666.

are not tangent to curves of N or $\bar{N}$, the focal nets of $\bar{G}$ are a second pair of corresponding nets on S and $\bar{S}$. Consequently the asymptotic lines correspond on S and $\bar{S}$,* and $\bar{G}$ is a $\bar{W}$ congruence.

If N is to be a derived net of $\bar{N}$, we must have

$$x = \bar{x} + \bar{p}\frac{\partial \bar{x}}{\partial u} + \bar{q}\frac{\partial \bar{x}}{\partial v}, \tag{32}$$

and the point equation of N analogous to (31) is

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial}{\partial v}\log ap\bar{p}\frac{\partial \theta}{\partial u} + \frac{\partial}{\partial u}\log bq\bar{q}\frac{\partial \theta}{\partial v}. \tag{33}$$

Comparing this equation with (1), we have $p\bar{p} = \bar{U}$, $q\bar{q} = \bar{V}$, where $\bar{U}$ and $\bar{V}$ are functions of u and v respectively. From (24) and (32) it is seen that the parameters u and v can be chosen so that

$$p\bar{p} = q\bar{q} = c', \tag{34}$$

where c' is a constant.

When the expression (24) for $\bar{x}$ is substituted in (32), the resulting equation is reducible to the form (2) where $r = -1$, and

$$\begin{aligned} -a' &= \frac{p}{c'} + \frac{1}{p} + \frac{1}{p}\frac{\partial p}{\partial u} + \left(\frac{q}{p} + \frac{p}{q}\right)\frac{\partial \log a}{\partial v} + \frac{1}{q}\frac{\partial p}{\partial v}, \\ -b' &= \frac{q}{c'} + \frac{1}{q} + \frac{1}{q}\frac{\partial q}{\partial v} + \left(\frac{p}{q} + \frac{q}{p}\right)\frac{\partial \log b}{\partial u} + \frac{1}{p}\frac{\partial q}{\partial u}. \end{aligned} \tag{35}$$

By means of (30) equations (35) are reducible to

$$\left(\frac{\partial^2\theta_1}{\partial v^2} + \frac{\partial^2\theta_1}{\partial u^2} - a'\frac{\partial\theta_1}{\partial u} + \frac{1}{c'}\theta_1\right)\frac{\partial\theta_2}{\partial v} = \left(\frac{\partial^2\theta_2}{\partial v^2} + \frac{\partial^2\theta_2}{\partial u^2} - a'\frac{\partial\theta_2}{\partial u} + \frac{1}{c'}\theta_2\right)\frac{\partial\theta_1}{\partial v},$$

$$\left(\frac{\partial^2\theta_1}{\partial v^2} + \frac{\partial^2\theta_1}{\partial u^2} - b'\frac{\partial\theta_1}{\partial v} + \frac{1}{c'}\theta_1\right)\frac{\partial\theta_2}{\partial u} = \left(\frac{\partial^2\theta_2}{\partial v^2} + \frac{\partial^2\theta_2}{\partial u^2} - b'\frac{\partial\theta_2}{\partial v} + \frac{1}{c'}\theta_2\right)\frac{\partial\theta_1}{\partial u},$$

from which it follows that θ_1 and θ_2 are solutions of

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = a'\frac{\partial\theta}{\partial u} + b'\frac{\partial\theta}{\partial v} + c'\theta. \tag{36}$$

In order that equations (1) and (2) with $r = -1$ may have three independent solutions, the functions P, Q and R in (9) must be equal to zero. Since θ_1 and θ_2 must be common solutions of (1) and (36), it follows then from (9) that $S = 0$. Hence from the last of (11) we have (17) and then from the first of (11) we get (18), that is N is a net R with equations reducible to (19). When these conditions are satisfied, so also are equa-

* E., p. 130.

tions (11) for (1) and (36), and consequently this system admits four independent solutions.

If we substitute in (24) the expression (32) for x, we find that in consequence of (34) and (35) the coördinates of $\bar{N}$ satisfy the equation

$$(37) \qquad \frac{\partial^2 \bar{\theta}}{\partial u^2} + \frac{\partial^2 \bar{\theta}}{\partial v^2} = 2\frac{\partial}{\partial u}\log ap\frac{\partial \bar{\theta}}{\partial u} + 2\frac{\partial}{\partial v}\log bq\frac{\partial \bar{\theta}}{\partial v}.$$

From (31) and (37) it follows that $\bar{N}$ is an R net.

We have just seen that if θ_1 and θ_2 are independent solutions of (1) and (36) the tangent planes to $\bar{N}$ pass through the corresponding points of N. In order that N be a derived net of $\bar{N}$ there must be two solutions of (31) such that $\bar{p}$ and $\bar{q}$ are of the forms (25). The latter are equivalent to the condition that θ_1 and θ_2 satisfy the relation

$$\theta + p\frac{\partial \theta}{\partial u} + q\frac{\partial \theta}{\partial v} = 0.$$

Hence in consequence of (34) there must exist two solutions of (31) such that

$$(38) \qquad \bar{\theta} + \frac{c'}{p}\frac{\partial \bar{\theta}}{\partial u} + \frac{c'}{q}\frac{\partial \bar{\theta}}{\partial v} = 0.$$

Differentiating this equation with respect to u and v and making use of (31) and (35) we obtain

$$\frac{\partial^2 \bar{\theta}}{\partial u^2} = \left(2\frac{\partial}{\partial u}\log ap + \frac{1}{p} + \frac{q}{p}\frac{\partial \log a}{\partial v}\right)\frac{\partial \bar{\theta}}{\partial u} - \frac{p}{q}\frac{\partial \log b}{\partial u}\frac{\partial \bar{\theta}}{\partial v},$$

$$\frac{\partial^2 \bar{\theta}}{\partial v^2} = -\frac{q}{p}\frac{\partial \log a}{\partial v}\frac{\partial \bar{\theta}}{\partial u} + \left(2\frac{\partial}{\partial v}\log bq + \frac{1}{q} + \frac{p}{q}\frac{\partial \log b}{\partial u}\right)\frac{\partial \bar{\theta}}{\partial v}.$$

By differentiation we find that these equations are consistent with (31), and consequently there exist two independent solutions of (31) and (38). Hence we have the theorem:

If N is a net R whose coördinates satisfy (19), *each pair of solutions of* (1) *and*

$$(39) \qquad \frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = 2\frac{\partial \log a}{\partial u}\frac{\partial \theta}{\partial u} + 2\frac{\partial \log b}{\partial v}\frac{\partial \theta}{\partial v} + c\theta,$$

where c is an arbitrary constant, determines an R net $\bar{N}$ without further quadratures such that N and $\bar{N}$ are derived nets of one another; and N and $\bar{N}$ are the focal nets of a W congruence.

Since equations (1) and (39) form a completely integrable system, any solution is expressible linearly in terms of four of them. Hence from the form of (25) we have the theorem:

An R net admits $\infty^5 W$ transforms into R nets for each value of c in (39).

6. **Theorem of permutability of transformations W.** Let θ_3 and θ_4 be two solutions of the equations

$$
(40)\qquad
\begin{aligned}
\frac{\partial^2\theta}{\partial u\partial v} &= \frac{\partial}{\partial v}\log a\cdot\frac{\partial\theta}{\partial u}+\frac{\partial}{\partial u}\log b\cdot\frac{\partial\theta}{\partial v},\\
\frac{\partial^2\theta}{\partial u^2}+\frac{\partial^2\theta}{\partial v^2} &= 2\frac{\partial\log a}{\partial u}\frac{\partial\theta}{\partial u}+2\frac{\partial\log b}{\partial v}\frac{\partial\theta}{\partial v}+c'\theta,
\end{aligned}
$$

where c' is a constant, and consider the functions

$$
(41)\qquad \bar\theta_i = \theta_i + p\frac{\partial\theta_i}{\partial u}+q\frac{\partial\theta_i}{\partial v}\qquad (i=3,4),
$$

where p and q are given by (25). Analogously to (26) we have

$$
(42)\qquad
\begin{aligned}
\frac{\partial\bar\theta_i}{\partial u} &= p\left\{\frac{\partial^2\theta_i}{\partial u^2}+\frac{1}{\Delta}\left[\frac{\partial\theta_i}{\partial u}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2}-\frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right)\right.\right.\\
&\qquad\left.\left.-\frac{\partial\theta_i}{\partial v}\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial u^2}-\frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial u^2}\right)\right]\right\}\\
\frac{\partial\bar\theta_i}{\partial v} &= q\left\{\frac{\partial^2\theta_i}{\partial v^2}+\frac{1}{\Delta}\left[\frac{\partial\theta_i}{\partial u}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial v^2}-\frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial v^2}\right)\right.\right.\\
&\qquad\left.\left.-\frac{\partial\theta_i}{\partial v}\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial v^2}-\frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial v^2}\right)\right]\right\}\\
&= q\left\{-\frac{\partial^2\theta_i}{\partial u^2}+(c'-c)\theta_i+c\bar\theta_i-\frac{1}{\Delta}\frac{\partial\theta_i}{\partial u}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2}-\frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right)\right.\\
&\qquad\left.+\frac{1}{\Delta}\frac{\partial\theta_i}{\partial v}\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial u^2}-\frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial u^2}\right)\right\}\\
&= q\left\{-\frac{1}{p}\frac{\partial\bar\theta_i}{\partial u}+(c'-c)\theta_i+c\bar\theta_i\right\}.
\end{aligned}
$$

Differentiating the first of these expressions with respect to u, we get

$$
\begin{aligned}
\frac{\partial^2\bar\theta_i}{\partial u^2} &= \frac{\partial\bar\theta_i}{\partial u}\left[\frac{\partial}{\partial u}\log pa^2b+\frac{1}{\Delta}\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2}-\frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right)\right]+p(c'-c)\frac{\partial\theta_i}{\partial u}\\
&\qquad -p\frac{\partial\log b}{\partial u}\left[c'\theta_i+c\left(p\frac{\partial\theta_i}{\partial u}+q\frac{\partial\theta_i}{\partial v}\right)\right].
\end{aligned}
$$

Making use of the expression (30) for $\partial p/\partial u$, we reduce this equation to

$$
\begin{aligned}
\frac{\partial^2\bar\theta_i}{\partial u^2} &= \frac{\partial\bar\theta_i}{\partial u}\left[\frac{\partial}{\partial u}\log p^2a^2b+\frac{1}{p}+\frac{q}{p}\frac{\partial\log a}{\partial v}\right]+p(c'-c)\frac{\partial\theta_i}{\partial u}\\
&\qquad -p\frac{\partial\log b}{\partial u}[(c'-c)\theta_i+c\bar\theta_i].
\end{aligned}
$$

Also we find that

$$\frac{\partial^2\bar{\theta}_i}{\partial u\partial v} = \frac{\partial}{\partial v}\log ap \cdot \frac{\partial\bar{\theta}_i}{\partial u} + \frac{\partial}{\partial u}\log bq \cdot \frac{\partial\bar{\theta}_i}{\partial v}.$$

From the second of (42) we get

$$\frac{\partial^2\bar{\theta}_i}{\partial v^2} = -\frac{q}{p}\frac{\partial \log a}{\partial v}\frac{\partial\bar{\theta}_i}{\partial u} + \left(\frac{\partial \log q}{\partial v} - \frac{q}{p}\frac{\partial}{\partial u}\log bq + cq\right)\frac{\partial\bar{\theta}_i}{\partial v} + q(c' - c)\frac{\partial\theta}{\partial v}.$$

With the aid of the expressions (30) we obtain

$$\frac{\partial^2\bar{\theta}_i}{\partial u^2} + \frac{\partial^2\bar{\theta}_i}{\partial v^2} = 2\frac{\partial}{\partial u}\log ap\frac{\partial\bar{\theta}_i}{\partial u} + 2\frac{\partial}{\partial v}\log bq\frac{\partial\bar{\theta}_i}{\partial v} + c'\bar{\theta}_i.$$

Hence the functions $\bar{\theta}_3$ and $\bar{\theta}_4$ determine a W transform of $\bar{N}$. The coordinates of this transform $\hat{N}$ are of the form

$$\hat{x} = \bar{x} + \bar{p}\frac{\partial\bar{x}}{\partial u} + \bar{q}\frac{\partial\bar{x}}{\partial v}, \tag{43}$$

where $\partial\bar{x}/\partial u$ and $\partial\bar{x}/\partial v$ are given by (26), and

$$\begin{gathered}\bar{p} = \frac{1}{\bar{\Delta}}\left(\bar{\theta}_3\frac{\partial\bar{\theta}_4}{\partial v} - \bar{\theta}_4\frac{\partial\bar{\theta}_3}{\partial v}\right), \qquad \bar{q} = \frac{1}{\bar{\Delta}}\left(\bar{\theta}_4\frac{\partial\bar{\theta}_3}{\partial u} - \bar{\theta}_3\frac{\partial\bar{\theta}_4}{\partial u}\right), \\ \bar{\Delta} = \frac{\partial\bar{\theta}_4}{\partial u}\frac{\partial\bar{\theta}_3}{\partial v} - \frac{\partial\bar{\theta}_3}{\partial u}\frac{\partial\bar{\theta}_4}{\partial v}.\end{gathered} \tag{44}$$

From the above expressions we find

$$\begin{aligned}\bar{\Delta} &= (c' - c)\left(\theta_3\frac{\partial\bar{\theta}_4}{\partial u} - \theta_4\frac{\partial\bar{\theta}_3}{\partial u}\right) + c\left(\bar{\theta}_3\frac{\partial\bar{\theta}_4}{\partial u} - \bar{\theta}_4\frac{\partial\bar{\theta}_3}{\partial u}\right) \\ &= \frac{p}{\Delta}\Bigg\{c'\Bigg[\left(\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_2}{\partial v}\frac{\partial\theta_1}{\partial u}\right)\left(\theta_3\frac{\partial^2\theta_4}{\partial u^2} - \theta_4\frac{\partial^2\theta_3}{\partial u^2}\right) \\ &\qquad + \left(\theta_3\frac{\partial\theta_4}{\partial u} - \theta_4\frac{\partial\theta_3}{\partial u}\right)\left(\frac{\partial\theta_2}{\partial v}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial v}\frac{\partial^2\theta_2}{\partial u^2}\right) \\ &\qquad - \left(\theta_3\frac{\partial\theta_4}{\partial v} - \theta_4\frac{\partial\theta_3}{\partial v}\right)\left(\frac{\partial\theta_2}{\partial u}\frac{\partial^2\theta_1}{\partial u^2} - \frac{\partial\theta_1}{\partial u}\frac{\partial^2\theta_2}{\partial u^2}\right)\Bigg] \\ &\qquad + c\Bigg[\left(\frac{\partial\theta_3}{\partial u}\frac{\partial\theta_4}{\partial v} - \frac{\partial\theta_4}{\partial u}\frac{\partial\theta_3}{\partial v}\right)\left(\theta_2\frac{\partial^2\theta_1}{\partial u^2} - \theta_1\frac{\partial^2\theta_2}{\partial u^2}\right) \\ &\qquad + \left(\theta_2\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta_2}{\partial u}\right)\left(\frac{\partial\theta_3}{\partial v}\frac{\partial^2\theta_4}{\partial u^2} - \frac{\partial\theta_4}{\partial v}\frac{\partial^2\theta_3}{\partial u^2}\right) \\ &\qquad - \left(\theta_2\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta_2}{\partial v}\right)\left(\frac{\partial\theta_3}{\partial u}\frac{\partial^2\theta_4}{\partial u^2} - \frac{\partial\theta_4}{\partial u}\frac{\partial^2\theta_3}{\partial u^2}\right)\Bigg]\Bigg\}.\end{aligned} \tag{45}$$

By making use of first of these expressions for $\bar{\Delta}$, (24) and (26), we reduce (43) to the form

$$\hat{x} = x + \frac{c' - c}{\Delta\bar{\Delta}} p \left\{ \frac{\partial x}{\partial u} \left[\left(\theta_1 \frac{\partial \theta_2}{\partial v} - \theta_2 \frac{\partial \theta_1}{\partial v} \right) \left(\theta_3 \frac{\partial^2 \theta_4}{\partial u^2} - \theta_4 \frac{\partial^2 \theta_3}{\partial u^2} \right) - \left(\theta_4 \frac{\partial \theta_3}{\partial v} - \theta_3 \frac{\partial \theta_4}{\partial v} \right) \left(\theta_2 \frac{\partial^2 \theta_1}{\partial u^2} - \theta_1 \frac{\partial^2 \theta_2}{\partial u^2} \right) \right] \right.$$
$$\left. - \frac{\partial x}{\partial v} \left[\left(\theta_1 \frac{\partial \theta_2}{\partial u} - \theta_2 \frac{\partial \theta_1}{\partial u} \right) \left(\theta_3 \frac{\partial^2 \theta_4}{\partial u^2} - \theta_4 \frac{\partial^2 \theta_3}{\partial u^2} \right) - \left(\theta_4 \frac{\partial \theta_3}{\partial u} - \theta_3 \frac{\partial \theta_4}{\partial u} \right) \left(\theta_2 \frac{\partial^2 \theta_1}{\partial u^2} - \theta_1 \frac{\partial^2 \theta_2}{\partial u^2} \right) \right] \right\}. \tag{46}$$

In consequence of (45) the expression (46) is symmetrical in c and c', and in the pairs of functions θ_1, θ_2 and θ_3, θ_4. Hence the net $\hat{N}$ can be obtained also by applying to N the W transformation determined by θ_3 and θ_4, and then to the resulting net $\bar{\bar{N}}$ the transformation determined by the functions

$$\bar{\theta}_i = \theta_i + \frac{\left(\theta_4 \frac{\partial \theta_3}{\partial v} - \theta_3 \frac{\partial \theta_4}{\partial v} \right) \frac{\partial \theta_i}{\partial u} - \left(\theta_4 \frac{\partial \theta_3}{\partial u} - \theta_3 \frac{\partial \theta_4}{\partial u} \right) \frac{\partial \theta_i}{\partial v}}{\frac{\partial \theta_3}{\partial u} \frac{\partial \theta_4}{\partial v} - \frac{\partial \theta_4}{\partial u} \frac{\partial \theta_3}{\partial v}} \qquad (i = 1, 2),$$

which are analogous to (41).

When $c' \neq c$, we have that $\hat{N}$ is a W transform of $\bar{N}$ and $\bar{\bar{N}}$. Hence:

If N is an R net, and N_1 and N_2 are obtained from N by transformations W_{c_1} and W_{c_2}, there can be found directly an R net N_{12}, which is in relations of transformations W_{c_2}' and W_{c_1}' with N_1 and N_2 respectively.

When $c' = c$, $\hat{N}$ coincides with N. Hence:

If an R net $\bar{N}$ is a W transform of an R net N by means of solutions θ_1 and θ_2 of equations (39), *and θ_3 and θ_4 are two other solutions of* (39) *independent of θ_1 and θ_2, then N is the W transform of $\bar{N}$ by means of the functions* (41).

This result is a proof of the theorem at the end of §5.

7. **Transformations** T. If N is any net whatsoever with its point equation of the form (1), and N' is a parallel net, defined by (20); if also θ and θ' are corresponding solutions of (1) and (22), that is in the relation (27), then equations of the form

$$x_1 = x - \frac{\theta}{\theta'} x' \tag{47}$$

give the coördinates of a net N_1 whose point equation is

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial}{\partial v} \log \frac{a\tau}{\theta'} \frac{\partial \theta}{\partial u} + \frac{\partial}{\partial u} \log \frac{b\sigma}{\theta'} \frac{\partial \theta}{\partial v}, \tag{48}$$

where

$$\tau = h\theta - \theta', \qquad \sigma = l\theta - \theta'. \tag{49}$$

The developables of the congruence G of lines joining corresponding points of N and N_1 meet the surfaces on which these nets lie in the curves of the nets.*

If N' and N'' are two nets parallel to N, defined by equations of the form

$$\frac{\partial x'}{\partial u} = h_1 \frac{\partial x}{\partial u}, \qquad \frac{\partial x'}{\partial v} = l_1 \frac{\partial x}{\partial v}; \qquad \frac{\partial x''}{\partial u} = h_2 \frac{\partial x}{\partial u}, \qquad \frac{\partial x''}{\partial v} = l_2 \frac{\partial x}{\partial v},$$

where h_1, l_1 and h_2, l_2 are pairs of solutions of (21), if θ_1 and θ_2 are solutions of (1) and θ_1' and θ_2'' are the corresponding solutions of the point equations of N' and N'', that is

$$\frac{\partial \theta_1'}{\partial u} = h_1 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = l_1 \frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta_2''}{\partial u} = h_2 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2''}{\partial v} = l_2 \frac{\partial \theta_2}{\partial v},$$

the equations of the form

$$x_1 = x - \frac{\theta_1}{\theta_1'} x', \qquad x_2 = x - \frac{\theta_2}{\theta_2''} x'' \tag{50}$$

define two T transforms, N_1 and N_2, of N. It is readily seen that the functions

$$\theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1'} \theta_2', \qquad \theta_{21} = \theta_1 - \frac{\theta_2}{\theta_2''} \theta_1'', \tag{51}$$

where

$$\frac{\partial \theta_2'}{\partial u} = h_1 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2'}{\partial v} = l_1 \frac{\partial \theta_2}{\partial v}; \qquad \frac{\partial \theta_1''}{\partial u} = h_2 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1''}{\partial v} = l_2 \frac{\partial \theta_1}{\partial v}, \tag{52}$$

are solutions of the point equations of N_1 and N_2 respectively.

Furthermore, equations of the forms

$$x_1''' = x'' - \frac{\theta_1''}{\theta_1'} x', \qquad x_2'''' = x' - \frac{\theta_2'}{\theta_2''} x'' \tag{53}$$

define nets N_1''' and N_2'''' parallel to N_1 and N_2 respectively. In fact,

$$\frac{\partial x_1'''}{\partial u} = h_{12} \frac{\partial x_1}{\partial u}, \qquad \frac{\partial x_1'''}{\partial v} = l_{12} \frac{\partial x_1}{\partial v}; \qquad \frac{\partial x_2''''}{\partial u} = h_{21} \frac{\partial x_2}{\partial u}, \qquad \frac{\partial x_2''''}{\partial v} = l_{21} \frac{\partial x_2}{\partial v},$$

* The results of this section are established in the memoir M. We say that G is the conjugate congruence of the transformation T defined by (47).

where

$$
(54)\qquad
\begin{aligned}
h_{12} &= \frac{h_1\theta_1'' - h_2\theta_1'}{\tau_1}, \qquad l_{12} = \frac{l_1\theta_1'' - l_2\theta_1'}{\sigma_1}, \\
&\qquad\qquad \tau_1 = h_1\theta_1 - \theta_1', \qquad \sigma_1 = l_1\theta_1 - \theta_1', \\
h_{21} &= \frac{h_2\theta_2' - h_1\theta_2''}{\tau_2}, \qquad l_{21} = \frac{l_2\theta_2' - l_1\theta_2''}{\sigma_2}, \\
&\qquad\qquad \tau_2 = h_2\theta_2 - \theta_2'', \qquad \sigma_2 = l_2\theta_2 - \theta_2''.
\end{aligned}
$$

Also if we write

$$
(55)\qquad \theta_{12}''' = \theta_2'' - \frac{\theta_1''}{\theta_1'}\theta_2', \qquad \theta_{21}'''' = \theta_1' - \frac{\theta_2'}{\theta_2''}\theta_1'',
$$

we have

$$
\frac{\partial\theta_{12}'''}{\partial u} = h_{12}\frac{\partial\theta_{12}}{\partial u}, \quad \frac{\partial\theta_{12}'''}{\partial v} = l_{12}\frac{\partial\theta_{12}}{\partial v}; \quad \frac{\partial\theta_{21}''''}{\partial u} = h_{21}\frac{\partial\theta_{21}}{\partial u}, \quad \frac{\partial\theta_{21}''''}{\partial v} = l_{21}\frac{\partial\theta_{21}}{\partial v}.
$$

It is readily shown that the following expressions for x_{12} are equal:

$$
x_{12} = x_1 - \frac{\theta_{12}}{\theta_{12}'''}x_1''' = x_2 - \frac{\theta_{21}}{\theta_{21}''''}x_2''''.
$$

These equations are of the form (50), and consequently x_{12}, y_{12}, z_{12} are the coördinates of a net N_{12} which is a T transfrom of N_1 and also of N_2. Since θ_2' and θ_1'' are determined only to within additive constants by (52), there are ∞^2 nets N_{12} which are T transforms of both N_1 and N_2.

8. **Transformations T of R nets.** Let N be an R net whose coördinates satisfy equations (19). A parallel net N' is defined by equations of the form

$$
(56)\qquad x' = \lambda\frac{\partial x}{\partial u} + \mu\frac{\partial x}{\partial v} + \nu\frac{\partial^2 x}{\partial u^2},
$$

where λ, μ and ν are determined by the condition that (20) hold. Making use of (5) we find that λ, μ and ν must satisfy the system of equations:

$$
(57)\qquad
\begin{aligned}
\frac{\partial\lambda}{\partial u} &= h - \mu\frac{\partial\log a}{\partial v} + \nu\left(\frac{1}{a}\frac{\partial^2 a}{\partial v^2} + 2\frac{\partial\log a}{\partial u}\frac{\partial\log b}{\partial u} - 2\frac{\partial^2\log a}{\partial u^2}\right. \\
&\qquad\qquad \left. - 2\frac{\partial\log b}{\partial v}\frac{\partial\log a}{\partial v}\right), \\
\frac{\partial\lambda}{\partial v} &= -\lambda\frac{\partial\log a}{\partial v} - 2\mu\frac{\partial\log a}{\partial u} - \nu\left(\frac{\partial^2\log a}{\partial u\partial v} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u}\right), \\
\frac{\partial\mu}{\partial u} &= -\mu\frac{\partial\log b}{\partial u} - \nu\left(\frac{\partial^2\log b}{\partial u\partial v} + \frac{\partial\log a}{\partial v}\frac{\partial\log b}{\partial u}\right), \\
\frac{\partial\mu}{\partial v} &= l - \lambda\frac{\partial\log b}{\partial u} - 2\mu\frac{\partial\log b}{\partial v} - \frac{\nu}{b}\frac{\partial^2 b}{\partial u^2}, \\
\frac{\partial\nu}{\partial u} &= -\lambda - \nu\frac{\partial}{\partial u}\log ba^2, \qquad \frac{\partial\nu}{\partial v} = \mu - \nu\frac{\partial\log a}{\partial v}.
\end{aligned}
$$

From (47) we have by differentiation

$$(58)\qquad \frac{\partial x_1}{\partial u} = \frac{\tau}{\theta'^2}\left(x'\frac{\partial \theta}{\partial u} - \theta'\frac{\partial x}{\partial u}\right), \qquad \frac{\partial x_1}{\partial v} = \frac{\sigma}{\theta'^2}\left(x'\frac{\partial \theta}{\partial v} - \theta'\frac{\partial x}{\partial v}\right),$$

and

$$\frac{\partial^2 x_1}{\partial u^2} = \frac{\partial}{\partial u}\log\frac{\tau}{\theta'^2}\frac{\partial x_1}{\partial u} + \frac{\tau}{\theta'^2}\left(x'\frac{\partial^2\theta}{\partial u^2} - \theta'\frac{\partial^2 x}{\partial u^2}\right),$$

$$\frac{\partial^2 x_1}{\partial v^2} = \frac{\partial}{\partial v}\log\frac{\sigma}{\theta'^2}\frac{\partial x_1}{\partial v} + \frac{\sigma}{\theta'^2}\left(x'\frac{\partial^2\theta}{\partial v^2} - \theta'\frac{\partial^2 x}{\partial v^2}\right).$$

By means of (56) and (58) these equations are reducible to

$$(59)\qquad \begin{aligned} \frac{\partial^2 x_1}{\partial u^2} &= \left(\frac{\partial}{\partial u}\log\frac{\tau}{\theta'^2} - \frac{\lambda}{\nu}\right)\frac{\partial x_1}{\partial u} - \frac{\mu}{\nu}\frac{\tau}{\sigma}\frac{\partial x_1}{\partial v} + \frac{\tau x'}{\theta'^2}\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\lambda}{\nu}\frac{\partial\theta}{\partial u}\right.\\ &\qquad\left. + \frac{\mu}{\nu}\frac{\partial\theta}{\partial v} - \frac{\theta'}{\nu}\right),\\ \frac{\partial^2 x_1}{\partial v^2} &= \frac{\sigma}{\tau}\left(2\frac{\partial\log a}{\partial u} + \frac{\lambda}{\nu}\right)\frac{\partial x_1}{\partial u} + \left(\frac{\partial}{\partial v}\log\frac{\sigma b^2}{\theta'^2} + \frac{\mu}{\nu}\right)\frac{\partial x_1}{\partial v}\\ &\quad + \frac{\sigma x'}{\theta'^2}\left[\frac{\partial^2\theta}{\partial v^2} - \left(\frac{\lambda}{\nu} + 2\frac{\partial\log a}{\partial u}\right)\frac{\partial\theta}{\partial u} - \left(\frac{\mu}{\nu} + 2\frac{\partial\log b}{\partial v}\right)\frac{\partial\theta}{\partial v} + \frac{\theta'}{\nu}\right]. \end{aligned}$$

From (19) and (48) it follows that N_1 will be an R net if

$$(60)\qquad \frac{\partial^2 x_1}{\partial u^2} + \frac{\partial^2 x_1}{\partial v^2} = 2\frac{\partial}{\partial u}\log\frac{a\tau}{\theta'}\frac{\partial x_1}{\partial u} + 2\frac{\partial}{\partial v}\log\frac{b\sigma}{\theta'}\frac{\partial x_1}{\partial v}.$$

Consequently we must have

$$(61)\qquad \begin{aligned} 2\frac{\partial}{\partial u}\log\frac{a\tau}{\theta'} &= \frac{\partial}{\partial u}\log\frac{\tau}{\theta'^2} - \frac{\lambda}{\nu} + \frac{\sigma}{\tau}\left(2\frac{\partial\log a}{\partial u} + \frac{\lambda}{\nu}\right),\\ 2\frac{\partial}{\partial v}\log\frac{b\sigma}{\theta'} &= \frac{\partial}{\partial v}\log\frac{\sigma b^2}{\theta'^2} + \frac{\mu}{\nu} - \frac{\tau}{\sigma}\frac{\mu}{\nu}, \end{aligned}$$

and

$$(62)\qquad \begin{aligned} &\frac{\partial^2\theta}{\partial u^2} + \frac{\lambda}{\nu}\frac{\partial\theta}{\partial u} + \frac{\mu}{\nu}\frac{\partial\theta}{\partial v} - \frac{\theta'}{\nu} = \frac{n\sigma}{\nu},\\ &\frac{\partial^2\theta}{\partial v^2} - \left(\frac{\lambda}{\nu} + 2\frac{\partial\log a}{\partial u}\right)\frac{\partial\theta}{\partial u} - \left(\frac{\mu}{\nu} + 2\frac{\partial\log b}{\partial v}\right)\frac{\partial\theta}{\partial v} + \frac{\theta'}{\nu} = -\frac{n\tau}{\nu}, \end{aligned}$$

where n is a function to be determined.

Equations (61) are reducible by means of (49) to

$$(63)\qquad \frac{\partial h}{\partial u} = (l - h)\left(\frac{\lambda}{\nu} + 2\frac{\partial\log a}{\partial u}\right), \qquad \frac{\partial l}{\partial v} = (l - h)\frac{\mu}{\nu}.$$

The function $(l-h)/\nu$ is found by differentiation to be a constant. Hence we have

$$(64)\qquad l - h = m\nu,$$

where m is a constant.

If the first and second of (62) are differentiated with respect to v and u respectively, we find that n is an arbitrary constant. Adding equations (62), we get (39) with

$$(65)\qquad c = mn.$$

Equations (21), (57) and (63) form a completely integrable system. If λ, μ, ν, h and l are multiplied by the same constant, the equations are satisfied and N' is replaced by a net homothetic to it, but the transform N_1 is the same. Hence there are four essential constants, and m is determined by (64). When one of these nets N' is known, each solution of (1) and (39) for a given c and n given by (65) determine the function θ' by means of (62). This transformation is unaltered if θ and θ' are multiplied by the same constant. Consequently the function θ involves only three essential constants in addition to c. Hence:

An R net N admits ∞^4 parallel nets determining congruences G of transformations T of N into R nets N_1; for each congruence there are ∞^4 of these nets N_1.

9. **Theorem of permutability of transformations T of R nets.** Let N_1 and N_2 be R nets obtained from an R net N by means of transformations T for which the functions are λ_i, μ_i, ν_i, h_i, l_i, m_i and n_i $(i = 1, 2)$. We desire to find the nets N_{12}, as defined in § 7, which are R nets.

From (51) we have by differentiation

$$\frac{\partial\theta_{12}}{\partial u} = \frac{\tau_1}{\theta_1'^2}\left(\theta_2'\frac{\partial\theta_1}{\partial u} - \theta_1'\frac{\partial\theta_2}{\partial u}\right), \qquad \frac{\partial\theta_{12}}{\partial v} = \frac{\sigma_1}{\theta_1'^2}\left(\theta_2'\frac{\partial\theta_1}{\partial v} - \theta_1'\frac{\partial\theta_2}{\partial v}\right)$$

and

$$\frac{\partial^2\theta_{12}}{\partial u^2} = \frac{\tau_1}{\theta_1'^2}\left[\theta_2'\frac{\partial^2\theta_1}{\partial u^2} - \theta_1'\frac{\partial^2\theta_2}{\partial u^2} + \frac{\partial}{\partial u}\log\frac{\tau_1}{\theta_1'^2}\left(\theta_2'\frac{\partial\theta_1}{\partial u} - \theta_1'\frac{\partial\theta_2}{\partial u}\right)\right],$$

$$\frac{\partial^2\theta_{12}}{\partial v^2} = \frac{\sigma_1}{\theta_1'^2}\left[\theta_2'\frac{\partial^2\theta_1}{\partial v^2} - \theta_1'\frac{\partial^2\theta_2}{\partial v^2} + \frac{\partial}{\partial v}\log\frac{\sigma_1}{\theta_1'^2}\left(\theta_2'\frac{\partial\theta_1}{\partial v} - \theta_1'\frac{\partial\theta_2}{\partial v}\right)\right].$$

From (60) it follows that the equation analogous to (39) is

$$(66)\qquad \frac{\partial^2\theta_{12}}{\partial u^2} + \frac{\partial^2\theta_{12}}{\partial v^2} = 2\frac{\partial}{\partial u}\log\frac{a\tau_1}{\theta_1'}\frac{\partial\theta_{12}}{\partial u} + 2\frac{\partial}{\partial v}\log\frac{b\sigma_1}{\theta_1'}\frac{\partial\theta_{12}}{\partial v} + c_1\theta_{12}.$$

If we substitute the above expressions and take

$$(67)\qquad c_1 = n_2m_2,$$

the resulting equation is reducible to

$$
\tag{68}
\begin{aligned}
m_1\left[\left(\lambda_1 - \lambda_2\frac{\nu_1}{\nu_2}\right)\frac{\partial\theta_2}{\partial u} + \left(\mu_1 - \mu_2\frac{\nu_1}{\nu_2}\right)\frac{\partial\theta_2}{\partial v} - \theta_2''\frac{\nu_1}{\nu_2}(n_2 - 1)\right]& \\
+ n_2\frac{\theta_2}{\nu_2}(l_1h_2 - l_2h_1) + (n_2m_2 - m_1)\theta_2' &= 0.
\end{aligned}
$$

When in like manner we require that

$$
\frac{\partial^2\theta_{21}}{\partial u^2} + \frac{\partial^2\theta_{21}}{\partial v^2} = 2\frac{\partial}{\partial u}\log\frac{a\tau_2}{\theta_2''}\frac{\partial\theta_{21}}{\partial u} + 2\frac{\partial}{\partial v}\log\frac{b\sigma_2}{\theta_2''}\frac{\partial\theta_{21}}{\partial v} + n_1m_1\theta_{21},
$$

we obtain

$$
\tag{69}
\begin{aligned}
m_2\left[\left(\lambda_2 - \lambda_1\frac{\nu_2}{\nu_1}\right)\frac{\partial\theta_1}{\partial u} + \left(\mu_2 - \mu_1\frac{\nu_2}{\nu_1}\right)\frac{\partial\theta_1}{\partial v} - \theta_1'\frac{\nu_2}{\nu_1}(n_1 - 1)\right]& \\
+ n_1\frac{\theta_1}{\nu_1}(l_2h_1 - l_1h_2) + (n_1m_1 - m_2)\theta_1'' &= 0.
\end{aligned}
$$

For the net N_1''', parallel to N_1, as given by (53), we have

$$
\tag{70}
x_1''' = \lambda_{12}\frac{\partial x_1}{\partial u} + \mu_{12}\frac{\partial x_1}{\partial v} + \nu_{12}\frac{\partial^2 x_1}{\partial u^2},
$$

and the functions h_{12} and l_{12} are given by (54). The equations analogous to (63) are

$$
\frac{\partial h_{12}}{\partial u} = (l_{12} - h_{12})\left(\frac{\lambda_{12}}{\nu_{12}} + 2\frac{\partial}{\partial u}\log\frac{a\tau_1}{\theta_1'}\right), \qquad \frac{\partial l_{12}}{\partial v} = (l_{12} - h_{12})\frac{\mu_{12}}{\nu_{12}}.
$$

On substituting the expressions (54) in these equations we get

$$
\tag{71}
\begin{aligned}
(l_{12} - h_{12})\left(\frac{\lambda_{12}}{\nu_{12}} + \frac{\partial}{\partial u}\log\frac{\tau_1}{\theta_1'^2}\right) &= \frac{\theta_1'}{\tau_1}\left(m_1\lambda_1\frac{l_2\theta_1 - \theta_1''}{\sigma_1} - m_2\lambda_2\right), \\
(l_{12} - h_{12})\frac{\mu_{12}}{\nu_{12}} &= \frac{\theta_1'}{\sigma_1}\left(m_1\mu_1\frac{l_2\theta_1 - \theta_1''}{\sigma_1} - m_2\mu_2\right).
\end{aligned}
$$

Equating the expressions (53) and (70) for x_1''', we get

$$
\begin{aligned}
\lambda_{12}\frac{\partial x_1}{\partial u} + \mu_{12}\frac{\partial x_1}{\partial v} + \nu_{12}\frac{\partial^2 x_1}{\partial u^2} &= \left(\lambda_2 - \frac{\theta_1''}{\theta_1'}\lambda_1\right)\frac{\partial x}{\partial u} + \left(\mu_2 - \frac{\theta_1''}{\theta_1'}\mu_1\right)\frac{\partial x}{\partial v} \\
&\quad + \left(\nu_2 - \frac{\theta_1''}{\theta_1'}\nu_1\right)\frac{\partial^2 x}{\partial u^2}.
\end{aligned}
$$

This equation is consistent with (71) when (69) is satisfied.

The equations analogous to (62) are

$$\frac{\partial^2\theta_{12}}{\partial u^2} + \frac{\lambda_{12}}{\nu_{12}}\frac{\partial\theta_{12}}{\partial u} + \frac{\mu_{12}}{\nu_{12}}\frac{\partial\theta_{12}}{\partial v} - \frac{\theta_{12}'''}{\nu_{12}} = \frac{n_2}{\nu_{12}}(l_{12}\theta_{12} - \theta_{12}'''),$$

$$\frac{\partial^2\theta_{12}}{\partial v^2} - \left(\frac{\lambda_{12}}{\nu_{12}} + 2\frac{\partial}{\partial u}\log\frac{a\tau_1}{\theta_1}\right)\frac{\partial\theta_{12}}{\partial u} - \left(\frac{\mu_{12}}{\nu_{12}} + 2\frac{\partial}{\partial v}\log\frac{b\sigma_1}{\theta_1}\right)\frac{\partial\theta_{12}}{\partial v} + \frac{\theta_{12}'''}{\nu_{12}}$$
$$= -\frac{n_2}{\nu_{12}}(h_{12}\theta_{12} - \theta_{12}''').$$

These equations are satisfied also when the above expressions are used and also (51) and (55). Hence the problem reduces to the determination of the conditions when (68) and (69) are satisfied. There are two cases to be considered, according as m_1n_1 and m_2n_2 are equal or not.

When $m_1n_1 = m_2n_2$, we have from equations of the form (62)

$$\left(\frac{\lambda_1}{\nu_1} - \frac{\lambda_2}{\nu_2}\right)\frac{\partial\theta_2}{\partial u} + \left(\frac{\mu_1}{\nu_1} - \frac{\mu_2}{\nu_2}\right)\frac{\partial\theta_2}{\partial v} - \frac{\theta_2'}{\nu_1} + \frac{\theta_2''}{\nu_2} = \frac{n_1}{\nu_1}(l_1\theta_2 - \theta_2')$$
$$- \frac{n_2}{\nu_2}(l_2\theta_2 - \theta_2''),$$

$$\left(\frac{\lambda_2}{\nu_2} - \frac{\lambda_1}{\nu_1}\right)\frac{\partial\theta_1}{\partial u} + \left(\frac{\mu_2}{\nu_2} - \frac{\mu_1}{\nu_1}\right)\frac{\partial\theta_1}{\partial v} - \frac{\theta_1''}{\nu_2} + \frac{\theta_1'}{\nu_1} = \frac{n_2}{\nu_2}(l_2\theta_1 - \theta_1'')$$
$$- \frac{n_1}{\nu_1}(l_1\theta_1 - \theta_1').$$

In consequence of these relations equations (68) and (69) are satisfied identically, and consequently each of the ∞^2 nets N_{12} is an R net. The determination of these nets requires the finding of θ_2' and θ_1'' from (52) by quadratures.

When $m_1n_1 \neq m_2n_2$, we find by differentiation that the left-hand members of (68) and (69) are constant. Hence when θ_2' and θ_1'' are given the expressions obtained from (68) and (69), the resulting N_{12} is an R net. Therefore we have the theorem:

If N is an R net, and N_1 and N_2 are two T transforms of N by means of functions θ_1 and θ_2 which are solutions of (1) *and* (39) *for the same value of c, all of the ∞^2 nets N_{12} are R nets, and their determination requires two quadratures; when the constant c in* (39) *is different for θ_1 and θ_2, there is a unique net N_{12} which is an R net and it can be found without quadrature.*

10. **Permutability of transformations W and T of R nets.** Let N be any net, $\bar{N}$ its derived net determined by solutions θ_1 and θ_2 of the point equation (1) of N, and N_3 the T transform of N defined by equations of the form

$$x_3 = x - \frac{\theta_3}{\theta_3'''}x''',$$

where θ_3 is a solution of (1), and $N'''(x''')$ is a net parallel to N, so that

$$\frac{\partial x'''}{\partial u} = h_3\frac{\partial x}{\partial u}, \qquad \frac{\partial x'''}{\partial v} = l_3\frac{\partial x}{\partial v}; \qquad \frac{\partial \theta_3'''}{\partial u} = h_3\frac{\partial \theta_3}{\partial u}, \qquad \frac{\partial \theta_3'''}{\partial v} = l_3\frac{\partial \theta_3}{\partial v}.$$

If θ_1''' and θ_2''' are defined by

$$(72) \quad \frac{\partial \theta_1'''}{\partial u} = h_3\frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1'''}{\partial v} = l_3\frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta_2'''}{\partial u} = h_3\frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2'''}{\partial v} = l_3\frac{\partial \theta_2}{\partial v},$$

the functions

$$(73) \qquad \theta_{31} = \theta_1 - \frac{\theta_3}{\theta_3'''}\theta_1''' \qquad \theta_{32} = \theta_2 - \frac{\theta_3}{\theta_3'''}\theta_2'''$$

are solutions of the point equation of N_3. They determine a derived net $\overline{N}_3$ of N_3 whose equations are of the form

$$(74) \qquad \bar{x}_3 = x_3 + p_3\frac{\partial x_3}{\partial u} + q_3\frac{\partial x_3}{\partial v},$$

where

$$p_3 = \frac{1}{\Delta_3}\left(\theta_{31}\frac{\partial \theta_{32}}{\partial v} - \theta_{32}\frac{\partial \theta_{31}}{\partial v}\right), \qquad q_3 = \frac{1}{\Delta_3}\left(\theta_{32}\frac{\partial \theta_{31}}{\partial u} - \theta_{31}\frac{\partial \theta_{32}}{\partial u}\right),$$

$$\Delta_3 = \frac{\partial \theta_{32}}{\partial u}\frac{\partial \theta_{31}}{\partial v} - \frac{\partial \theta_{32}}{\partial v}\frac{\partial \theta_{31}}{\partial u}.$$

From (73) we have

$$\left.\begin{aligned} \frac{\partial \theta_{3i}}{\partial u} &= \frac{\theta_3 h_3 - \theta_3'''}{\theta_3'''^2}\left(\theta_i'''\frac{\partial \theta_3}{\partial u} - \theta_3'''\frac{\partial \theta_i}{\partial u}\right), \\ \frac{\partial \theta_{3i}}{\partial v} &= \frac{\theta_3 l_3 - \theta_3'''}{\theta_3'''^2}\left(\theta_i'''\frac{\partial \theta_3}{\partial v} - \theta_3'''\frac{\partial \theta_i}{\partial v}\right). \end{aligned}\right\} \qquad (i = 1, 2).$$

Also we have

$$\frac{\partial x_3}{\partial u} = \frac{\theta_3 h_3 - \theta_3'''}{\theta_3'''^2}\left(x'''\frac{\partial \theta_3}{\partial u} - \theta_3'''\frac{\partial x}{\partial u}\right),$$

$$\frac{\partial x_3}{\partial v} = \frac{\theta_3 l_3 - \theta_3'''}{\theta_3'''^2}\left(x'''\frac{\partial \theta_3}{\partial v} - \theta_3'''\frac{\partial x}{\partial v}\right).$$

On substituting these expressions in (74), the resulting equation is reducible to

$$
\begin{aligned}
\bar{x}_3 = x + \frac{1}{\Delta'}\Bigg\{ x'''\bigg[\theta_1\left(\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_3}{\partial v} - \frac{\partial\theta_3}{\partial u}\frac{\partial\theta_2}{\partial v}\right) \\
+ \theta_2\left(\frac{\partial\theta_3}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_1}{\partial u}\frac{\partial\theta_3}{\partial v}\right) + \theta_3\left(\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v} - \frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v}\right)\bigg] \\
+ \frac{\partial x}{\partial u}\bigg[(\theta_2\theta_1''' - \theta_1\theta_2''')\frac{\partial\theta_3}{\partial v} + (\theta_3\theta_2''' - \theta_2\theta_3''')\frac{\partial\theta_1}{\partial v} \\
+ (\theta_1\theta_3''' - \theta_3\theta_1''')\frac{\partial\theta_2}{\partial v}\bigg] - \frac{\partial x}{\partial v}\bigg[(\theta_2\theta_1''' - \theta_1\theta_2''')\frac{\partial\theta_3}{\partial u} \\
+ (\theta_3\theta_2'' - \theta_2\theta_3''')\frac{\partial\theta_1}{\partial u} + (\theta_1\theta_3''' - \theta_3\theta_1''')\frac{\partial\theta_2}{\partial u}\bigg]\Bigg\},
\end{aligned} \tag{75}
$$

where

$$
\begin{aligned}
\Delta' = \theta_2'''\left(\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_3}{\partial v} - \frac{\partial\theta_1}{\partial v}\frac{\partial\theta_3}{\partial u}\right) + \theta_1'''\left(\frac{\partial\theta_2}{\partial v}\frac{\partial\theta_3}{\partial u} - \frac{\partial\theta_2}{\partial u}\frac{\partial\theta_3}{\partial v}\right) \\
+ \theta_3'''\left(\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_2}{\partial v}\frac{\partial\theta_1}{\partial u}\right).
\end{aligned}
$$

In § 4 it is shown that the equations of the form

$$\bar{x}''' = x''' + p'''\frac{\partial x'''}{\partial u} + q'''\frac{\partial x'''}{\partial v},$$

where

$$p''' = \theta_1'''\frac{\partial\theta_2'''}{\partial v} - \theta_2'''\frac{\partial\theta_1'''}{\partial v}, \qquad q''' = \theta_2'''\frac{\partial\theta_1'''}{\partial u} - \theta_1'''\frac{\partial\theta_2'''}{\partial u},$$

$$\Delta''' = \left(\frac{\partial\theta_2'''}{\partial u}\frac{\partial\theta_1'''}{\partial v} - \frac{\partial\theta_1'''}{\partial v}\frac{\partial\theta_2'''}{\partial u}\right),$$

define a net $\bar{N}'''$ parallel to $\bar{N}$. The functions

$$\bar{\theta}_3 = \theta_3 + p\frac{\partial\theta_3}{\partial u} + q\frac{\partial\theta_3}{\partial v}, \qquad \bar{\theta}_3''' = \theta_3''' + p'''\frac{\partial\theta_3'''}{\partial u} + q'''\frac{\partial\theta_3'''}{\partial v}$$

are corresponding solutions of the point equations of $\bar{N}$ and $\bar{N}'''$. Hence expressions of the form

$$\bar{x} - \frac{\bar{\theta}_3}{\bar{\theta}_3'''}\bar{x}'''$$

determine a T transform of $\bar{N}$. When the above expressions for these functions are substituted, the result is reducible to the right-hand member of (75). Since θ_1''' and θ_2''', as defined by (72) involve additive arbitrary constants, there are ∞^2 of the nets $\bar{N}_3$. Hence we have the theorem:

If $\bar{N}$ is a derived net of any net N_1 and N_3 is any T transform of N,

there can be found by two quadratures ∞^2 *nets* $\overline{N}_3$, *each of which is a derived net of* N_3 *and a* T *transform of* $\overline{N}$.

We apply this theorem true for any net to the case when N is an R net. Let $\overline{N}$ be the R net obtained from N by two solutions θ_1 and θ_2 of equations (1) and (39), and let N_3 be a T transform of N in accordance with the results of § 8. From equations analogous to those of § 9 it follows that the functions (73) satisfy equations for N_3 analogous to (39). Moreover, if θ_3 is a solution of (39), then θ_1''' and θ_2''' are determined by quadratures and involve additive constants; but if θ_3 is a solution of the equation obtained by replacing c by c' in (39), then θ_1''' and θ_2''' are uniquely determined. Hence:

If $\overline{N}$ *is a* W *transform of an* R *net* N *by means of solutions* θ_1 *and* θ_2 *of* (1) *and* (39), *and* N_3 *is an* R *net which is a* T *transform by means of a function* θ_3, *a solution of* (1) *and* (39) *with* c *replaced by* c', *there can be found directly a unique* R *net* $\overline{N}_3$ *which is a* W *transform of* N_3 *and a* T *transform of* $\overline{N}$; *when* $c' = c$, *there are* ∞^2 *such nets* $\overline{N}_3$, *obtained by two quadratures.*

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